There's a bear in there

Our Maths Masters finish the year on a puzzling note. By **Burkard Polster** and **Marty Ross**.

HERE is a famous puzzle: A bear leaves his den. He travels 10 kilometres south, then 10 kilometres east, and finally 10 kilometres north. He finds that he is back in his den. What colour is the bear?

This is a great puzzle for so many reasons. First of all, it's funny and engaging. What can maths possibly say about the colour of a bear?

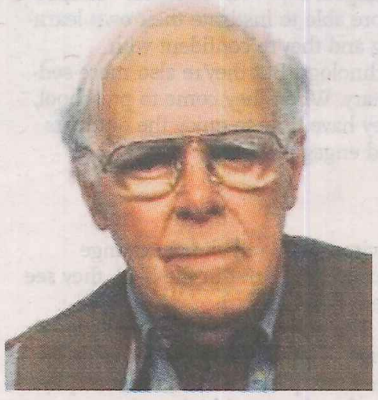
Second, it's a genuinely puzzling puzzle. At first glance the scenario seems impossible, since compass directions are all about right angles. Surely, we just have three sides of a square?

Third, the solution (which we give below) is delightfully simple. And, the puzzle offers a sneaky double-punch. Even many people who know this puzzle are surprised to learn there is more than one solution.

Fourth, the puzzle provides an engaging entry to beautiful and deep mathematics. As we shall try to make clear this is no accident: at heart mathematicians are simply happy children, being paid to solve puzzles.

Mathematical games

Undoubtedly, the greatest mathematics populariser of all time is Martin Gardner. His wonderful Mathematical Games column ran in *Scientific American* for 25 years. Through it, Gardner introduced non-mathematicians to a wealth of beautiful mathematics, and always with clarity, humour and elegance. At 94, Gardner is still writing his beautiful books.



Martin Gardner

Gardner's columns have been collected in a series of books, beginning in 1959 with *Mathematical Puzzles and Diversions*. Our bear problem above is Gardner's very first puzzle in this very first collection.

In his exposition of the puzzle, Gardner begins with the well-known solution: the bear is white. Why? Because the bear is a polar bear, with his den at the North Pole. So, he travels south from the North Pole (it doesn't matter along which line of longitude), then east along a circle of latitude. And, on the last leg he travels north, back to the North Pole.

A bear travels along a surprising triangular path.

After presenting this solution, Gardner challenges the reader to find another solution. Even with the proffered solution in mind, it is not easy to think of another. In fact, there are many!

Here is the idea for a new solution. The "bear" starts very near the South Pole: the bear is now a penguin! If he starts at just the right latitude, then going south the 10 kilometres will take him so close to the South Pole that traveling 10 kilometres east will take him on a full loop of the circle of latitude. Then, going north exactly retraces the first leg, and he ends up back where he started. We'll leave as homework for you to sort out the details, and to complete the puzzle: there are yet more solutions.

A South Pole solution for the "bear's" journey.

Bears in space

The bear puzzle is fun, with a beautiful solution. But this puzzle also points to a serious geographical problem. We like to think of the compass directions as providing natural rectangular co-ordinates. But, the bear's paths show that this cannot actually be the case. The geometry of the Earth must be fundamentally different to that of a flat plane.

This issue was of huge historical importance, for sailors wanting accurate maps to aid their navigation. The bear puzzle shows that any such map of the Earth's surface must unavoidably be distorted. The question then becomes, what's the least troubling distortion? This is a fascinating story, culminating in the 16th Century with (but my no means ended by) the map projection of Gerardus Mercator.

We shall use the bear puzzle as an excuse to ask a similar question in a wholly different context: what shape is space? That this question can even be asked often surprises people. Isn't space just space-shaped? There's an X direction and a Y direction and a Z direction, and that's it? Well, maybe not.

Imagine we place our bear (now an astronaut) into a rocket and shoot him out into space. He travels a million kilometres, takes a right angle and travels another million kilometres. He then takes another right angle and travels a final million kilometres.

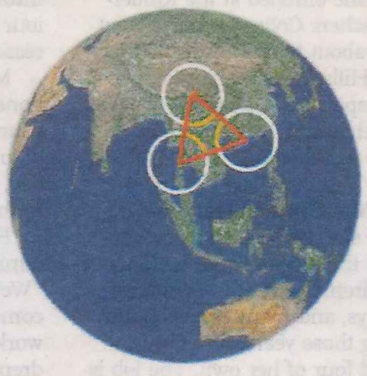
Could our bear possibly be back on Earth? Yes, depending upon the shape of our universe, he just might. It is possible that the universe is fundamentally curved, just like the surface of the Earth, and that the rectangular X-Y-Z "compass" directions simply don't work.

Mathematically, the way to determine whether the universe is curved is exactly by shooting (theoretical) bears into space and considering their triangular paths. The amount the universe is curved is then indicated by the sum of the angles in the triangles. In a rectangular Euclidean world the sum of the angles must be 180 degrees. But, in other imaginable worlds, and perhaps in our actual universe, the sum may be greater or smaller.

Torturing triangles

This is exactly what happens on the surface of the Earth. The sum of the angles in a spherical triangle is always greater than 180 degrees. How much greater? The precise answer is given by the following beautiful formula. If the Earth has area E and the triangle has area A, then the sum of the angles is

$$S = 180^\circ + \frac{720^\circ A}{E}$$



We can picture the three legs as either making a very small triangle with orange angles close to 60°, or a very large triangle with white angles close to 300°.

So, a tiny triangle has a sum very close to 180 degrees. But if the triangle grows to consume almost all of the Earth then A will be very close to E, and so the sum approaches 180+720=900 degrees!

What does such a large triangle look like? Exactly like the tiny equilateral triangle, except we declare the "inside" to be the "outside" and vice versa. That is, instead of three angles of a little more than 60 degrees (the orange arcs pictured), we picture the triangle as having three (white) angles each a little less than 300 degrees!

Notice that there is another, intriguing way to view the above formula. Suppose we take a small triangle on the Earth, and we measure the triangle's area and the sum of its angles. Then, the formula can be rearranged to tell us the Earth's area. That is, as long as we assume the Earth is uniformly spherical, a little triangle exactly tells us the radius of that sphere. We have obtained global information from a tiny portion of the sphere.

And what about space? It's complicated – there's that whole fourth dimenion to worry about. But it is possible that the universe is curved and finite, exactly as is the Earth. And in principle we analyse the universe in exactly the same manner. At its heart it is the same as the bear puzzle: what's the sum of the angles in a triangle?

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ANSWERS TO LAST WEEK'S MATHS CHALLENGE

ONE POINT QUESTIONS

1. 8.
2. About 91 minutes.
3. Use a mirror.
4. 24.
5. About 16 centimeters.
6. 9/8.
7. A loss of $800.
8. 35.
9. $4,283,508,449.71.
10. 26 meters.
11. Pythagoras.
12. Maybe at the North Pole or South Pole, but not otherwise.
13. 2 + 2/2 + 2/2, 2 + 2 + 2 - 2/2, 2 + 2 + 2 + 2 - 2, 2 × 2 × 2 - 2/2, 2 × 2 × 2 + 2 - 2, 2 × 2 × 2 + 2/2 .
14. 2 and 8/3.
15. Safe, deadly, deadly.
16. 1 7/8.
17. 1,000,000.
18. 99.96%.
19. Plant at the corners of a tetrahedral mound.
20. 31.

TWO POINT QUESTIONS

21. √2 to 1. A4 paper and other standard paper sizes have these proportions.
22. It's a magic square, and is also a magic square upside down. In both cases all rows, columns and diagonals sum to 264.
23. All are fractions with denominator
24. 2.
25. Fill 5-Jug, move 3 gallons to 3-Jug and empty 3-Jug. Move remaining 2 gallons from 5-Jug to 3-Jug. Fill 5-Jug, and use 1 gallon to fill 3-Jug. The second problem is impossible.
26. Every symbol is a digit and its mirror image glued together. Next comes 8.
27. The same.
28. 49.
29. Start both. When the 4-glass stops, turn it over. When the 7-glass stops, there is one minute left in the 4-glass. Turn the 7-glass over. When the 4-glass runs out, 8 minutes have passed and 1 minute has run in the 7-glass. Turn the 7-glass over.
30. Form a 3-4-5 triangle.
31. 1/3.
32. If there are an even number of squares then the person who breaks first wins, and otherwise the second person wins.
33. Queen of Hearts, Queen of Spades, and King of Spades.
34. (1 + √5)/2, the golden ratio.
35. 15 kilometers per hour.
36. All except the regular pentagon are possible.
37. 1/2.
38. 1.
39. Set the corner of the book at a point on the perimeter of the circle, and draw a right angle using the edges of the book, and see where the lines intersect the circle.
40. 22/7 (anyone who answered π fails immediately!).

THREE POINT QUESTIONS

41. The trip is possible for the king and queen, and impossible for the others.
42. One mile.
43. 2/5.
44. The trip is impossible.
45. 89.
46. 18.
47. About 25 students.
48. 2, 2 and 9.
49. 20 squares, no equilateral triangles.
50. 44.
51. All are possible.
52. 32 8/11 minutes after midnight. It is impossible for the three hands to be equally spaced.
53. 6 different lacings. The first diagrammed lacing is the shortest.
54. The amounts level off to 2.71828 ...., known as Euler's number e.
55. 1/6 for the octahedron, and 1/3 for the tetrahedron.
56. 1321232116.
57. A skull.
58. 1 1/5 meters.
59. Yes.
60. The pink lens on the left has area π/2 - 1. The yellow curvy square on the right has area 1 - √3 + π/3.

Extended answers at www.qedcat.com